REVIEW SHEET 1 ON SERIES

1) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n3^n}$ $\quad \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x-5)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-5)}{3} \cdot \frac{n}{n+1} \right|$

$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \left| \frac{x-5}{3} \right| = 1 \cdot \left| \frac{x-5}{3} \right| < 1 \qquad |x-5| < 3$ $\boxed{\text{Radius of Convergence} = 3}$

$-3 < x-5 < 3$

Interval of Convergence $\nearrow$ $\boxed{2 \le x < 8}$ Test Endpoints: if $x=2$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n}$ = alternating harmonic series $\to$ converges
① alt ② dec. ③ lim = 0

if $x=8$: $\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum \frac{1}{n}$ harmonic $\to$ Diverges

② $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n!}$ $\quad \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{x-2}{n} \right| = 0 |x-2| < 1$

always < 1 for any x

Radius of Converges $= \infty$

Interval of Convergence $= (-\infty, \infty)$ (for all x or all real #s)

③ Find sum of $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots + \frac{(-1)^n}{(2n+1)!} + \dots$

Memorize that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$

Given series is same as $\sin x$ if $x=1$, so
given sum $= \sin(1)$.

④ $\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = 3 + \frac{3^2}{4} + \frac{3^3}{4^2} \dots$ geometric w/ $a_1 = 3$ & $r = \frac{3}{4}$

$\therefore$ Sum $= \frac{a_1}{1-r} = \frac{3}{1-\frac{3}{4}} = \frac{3}{\frac{1}{4}} = \boxed{12}$

⑤ $x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots = f(x)$ same as $\sin x$ if each term is multiplied by x, so $f(x) = x \sin x$

$f'(x) =$

⑥ $\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!}$

$f(x) = x - \frac{x^7}{2!\cdot 7} + \dots$ $\quad x^7 \text{ term} = -\frac{1}{14}x^7$

coefficient $= \boxed{-\frac{1}{14}}$

⑦ $\underbrace{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \frac{(-1)^n x^n}{n!} + \dots}_{= e^{-x}}$ intersects $y = x^3$ at $x = ?$

$= e^{-x}$ (same as e^x with $-x$ substituted for x)

Graph $y = e^{-x}$ and $y = x^3$ & find intersection: $\boxed{x = .773}$

⑧ $f(x) = \frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \dots + \frac{n+1}{4^{n+1}}x^n + \dots$ for all x in interval of conv.

ⓐ Find interval of convergence:

$$\lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{4^{n+2}}\cdot\frac{4^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+1}\cdot\frac{x}{4}\right| = 1\cdot\left|\frac{x}{4}\right| < 1$$

$|x| < 4$

Test Endpoints: $\boxed{-4 < x < 4}$

if $x = -4$ $\quad \sum \frac{(n+1)(-4)^n}{4^{n+1}} = \sum \frac{(n+1)(-1)^n \cancel{4^n}}{4\cdot\cancel{4^n}}$ $\quad \lim_{n\to\infty}\frac{n+1}{4} = \infty$ Diverges by nth term test

if $x = 4$ $\quad \sum \frac{(n+1)4^n}{4^{n+1}} = \sum \frac{(n+1)\cancel{4^n}}{4\cdot\cancel{4^n}}$

ⓑ $\lim_{x\to 0}\frac{f(x) - \frac{1}{4}}{x} = \lim_{x\to 0}\frac{\left(\cancel{\frac{1}{4}} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \dots + \frac{n+1}{4^{n+1}}x^n + \dots\right) - \cancel{\frac{1}{4}}}{x} =$

$= \lim_{x\to 0}\frac{2}{4^2} + \frac{3}{4^3}x + \dots + \frac{n+1}{4^{n+1}}x^{n-1} = \frac{2}{4^2} = \boxed{\frac{2}{16}}$

ⓒ $\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \dots + \frac{n+1}{4^{n+1}}x^n + \dots dx$

$= \left(\frac{1}{4}x + \frac{\cancel{2}}{\cancel{2}\cdot 4^2}x^2 + \frac{\cancel{3}}{\cancel{3}\cdot 4^3}x^3 + \dots \frac{\cancel{(n+1)}}{\cancel{(n+1)}4^{n+1}}x^{n+1}\right)\Big|_0^1$

$= \boxed{\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots + \frac{1}{4^{n+1}} + \dots}$

ⓓ geometric $a_1 = \frac{1}{4}$ $r = \frac{1}{4}$ $\quad S = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4}\cdot\frac{4}{3} = \boxed{\frac{1}{3}}$

9) $f(2)=3 \quad f'(2)=-4 \quad f''(2)=-1 \quad f'''(2)=5$

a) 3^{rd} degree Taylor Poly. @ $x=2$ and approximate $f(1.2)$

$$3-4(x-2)-\frac{(x-2)^2}{2!}+\frac{5(x-2)^3}{3!}$$

$$f(1.2)\approx 3-4(1.2-2)-\frac{(1.2-2)^2}{2}+\frac{5(1.2-2)^3}{6}=\boxed{5.453}$$

b) $$|R_3x| = \left|\frac{f^4(z)(x-2)^4}{4!}\right|$$

$$\le \frac{3(1.2-2)^4}{4!} = .0512 = \text{Max error using } P_3(x)$$

$$\therefore |f(1.2)-P_3(1.2)| \le .0512$$

$$|f(1.2)-5.453| \le .0512$$

f(2) MUST be on interval: $5.402 \le f(1.2) \le 5.505$

$\therefore f(1.2) \neq 5.3$ b/c 5.3 is outside above interval

c) $g(x)=f(x^2+2)$

$$g(x)\approx P_4(x)=3-4(x^2+2-2)-\frac{(x^2+2-2)^2}{2}+\frac{5(x^2+2-2)^3}{3!}+\frac{2(x^2+2-2)}{4!}$$

$$=3-4x^2-\frac{x^4}{2}\cancel{+\frac{5x^6}{3!}+\frac{2x^8}{4!}}$$ 4th degree, so only need 3 term

$$g'(x)\approx -8x-\frac{4x^3}{2}$$ @ $x=0$, $P'(x)=0$

$$g''(x)\approx -8-\frac{12x^2}{2}$$ @ $x=0$, $P''(x)<0$

$\therefore g(x)$ has relative max at $x=0$ by the 2^{nd} Deriv. Test

Review Sheet 2 on Series

① $f(x)=\cos(2x)$

Substitute $2x$ for x into Maclaurin Series for $\cos(x)$...

$$f(x)\approx 1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}=1-2x^2+\frac{2x^4}{3}-$$

$\boxed{C}$

② $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n(-3)^n}$ $\quad \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)(-3)^{n+1}}\cdot\frac{n(-3)^n}{(x-2)^n}\right|$

$$=\lim_{n\to\infty}\left|\frac{x-2}{-3}\cdot\frac{n}{n+1}\right|=\left|\frac{x-2}{-3}\right|<1 \quad |x-2|<3$$

$-3<x-2<3$

Test Endpoints: if $x=-1$: $\sum\frac{\cancel{(-3)^n}}{n\cancel{(-3)^n}}=\sum\frac{1}{n}$ diverges $\boxed{-1<x\le 5}$

$\boxed{C}$

if $x=5$: $\sum\frac{3^n}{n(-3)^n}=\sum\frac{1}{n(-1)^n}$ Alt. harmonic converges by Alt Series Test

① alt ② dec. ③ lim = 0

③ $x+x^3+\frac{x^5}{2!}+\frac{x^7}{3!}+\dots+\frac{x^{2n+1}}{n!}$ Maclaurin Series for...

all positive terms & not skipping denominator factorials, so how does this compare to $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots+\frac{x^n}{n!}+\dots$?

$\boxed{D}$ → substitue x^2 for each x above, then multiply each term by another x.

④ x^3 coefficient for Taylor series e^{2x} @ $x=0$ is...

$$e^{2x}=1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\dots$$

$$\frac{8x^3}{6}=\boxed{\frac{4}{3}}x^3 \quad \boxed{D}$$

⑤ $f(x)=\sum_{n=1}^{\infty}(\cos x)^{3n}$ Evaluate $f\left(\frac{2\pi}{3}\right)$ ⇒ equals sum of series when $x=\frac{2\pi}{3}$

$$\sum_{n=1}^{\infty}(\cos x)^{3n}=(\cos x)^3+(\cos x)^6+(\cos x)^9+\dots+(\cos x)^{3n}$$

$$=\left(-\tfrac{1}{2}\right)^3+\left(-\tfrac{1}{2}\right)^6+\left(-\tfrac{1}{2}\right)^9+\dots$$

$\cos\frac{2\pi}{3}=-\frac{1}{2}$

$$=-\frac{1}{8}+\frac{1}{64}-\frac{1}{512}$$ geometric series $a_1=-\frac{1}{8}$ $r=-\frac{1}{8}$

$$S=\frac{-\frac{1}{8}}{1-\left(-\frac{1}{8}\right)}=-\frac{1}{8}\cdot\frac{8}{9}=\boxed{-\frac{1}{9}}$$

⑥ Sum of $\frac{9}{8} - \frac{3}{4} + \frac{1}{2} - \frac{1}{3} + \dots$

geometric $a_1 = \frac{9}{8}$ $r = -\frac{2}{3}$

$$S = \frac{9/8}{1-(-2/3)} = \frac{9/8}{1+2/3} = \frac{9/8}{5/3} = \frac{9}{8} \cdot \frac{3}{5} = \boxed{\frac{27}{40}}$$

⑦ $f(x) = \sqrt{1+x} = (1+x)^{1/2}$ Taylor Poly, order 3 at $x=0$ $f(0)=1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $f'''(0) = \frac{3}{8}$

$$1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2} + \frac{3}{8}\frac{x^3}{3!} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

⑧ nth deriv. of f at $x=2$: $f^n(2) = \frac{(n+1)!}{3^n}$ for $n \geq 1$, $f(2)=1$

ⓐ $1 + \frac{2!}{3}(x-2) + \frac{3!}{3^2}\frac{(x-2)^2}{2!} + \frac{4!}{3^3}\frac{(x-2)^3}{3!} + \dots + \frac{(n+1)!}{3^n \cdot n!}(x-2)^n$

$= \frac{(n+1)(x-2)^n}{3^n} + \dots$

ⓑ $\lim_{n\to\infty} \left| \frac{(n+2)(x-2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x-2}{3} \cdot \frac{n+2}{n+1} \right| = \left| \frac{x-2}{3} \right| < 1$ $|x-2| < 3 \rightarrow$ Radius of Convergence $= 3$

ⓒ $g(2) = 3$ $g'(x) = f(x)$

$g(x) = \int g'(x)\,dx = x + \frac{(x-2)^2}{3} + \frac{(x-2)^3}{3^2} + \frac{(x-2)^4}{3^3} + \dots + \frac{(x-2)^{n+1}}{3^n} + \dots + C$

$g(2) = 2 + C$

$3 = 2 + C$ $C = 1$

$$g(x) = 1 + x + \frac{(x-2)^2}{3} + \frac{(x-2)^3}{3^2} + \dots + \frac{(x-2)^{n+1}}{3^n} + \dots$$

ⓓ $f(x)$ int of convergence: $-3 < x-2 < 3 \rightarrow -1 < x < 5$

Don't need to check endpoints b/c integrating to get $g(x)$ will result in same open interval, so NO $g(x)$ doesn't converge @ $x=-2$ b/c -2 is NOT in int. of conv.

(9) (a)

$f(x) = \cos\left(3x + \frac{3\pi}{4}\right) \quad f(0) = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$f'(x) = -3\sin\left(3x + \frac{3\pi}{4}\right) \quad f'(0) = -3\sin\frac{3\pi}{4} = -\frac{3\sqrt{2}}{2}$

$f''(x) = -9\cos\left(3x + \frac{3\pi}{4}\right) \quad f''(0) = -9\cos\frac{3\pi}{4} = \frac{9\sqrt{2}}{2}$

$f'''(x) = 27\sin\left(3x + \frac{3\pi}{4}\right) \quad f'''(0) = 27\sin\frac{3\pi}{2} = \frac{27\sqrt{2}}{2}$

$f^{4}(x) = 81\cos\left(3x + \frac{3\pi}{4}\right)$

$-\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}x + \frac{9\sqrt{2}}{2}\cdot\frac{x^2}{2} + \frac{27\sqrt{2}}{2\cdot 3!}x^3$

(b) $\left|f\left(\frac{1}{6}\right) - P\left(\frac{1}{6}\right)\right| \leq \left|\frac{f^{4}(z)\,x^4}{4!}\right| = \frac{81\left(\frac{1}{6}\right)^4}{4!} = \frac{81}{4\cdot 6^5} = \frac{81}{4\cdot 3^5\cdot 2^5}$

$= \frac{1}{4\cdot 3\cdot 2^5} \leq \frac{1}{300}$

(c) $G(x) = \int_0^x f(t)\,dt \quad G'(x) = f(x)$

$(x) = \int_0^x -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}t + \frac{9\sqrt{2}}{2}\frac{t^2}{2}\,dt = -\frac{\sqrt{2}}{2}x - \frac{3\sqrt{2}}{2}\frac{x^2}{2} + \frac{9\sqrt{2}}{2}\frac{x^3}{2\cdot 3}$

$= \boxed{-\frac{\sqrt{2}}{2}x - \frac{3\sqrt{2}}{4}x^2 + \frac{3\sqrt{2}}{4}x^3}$