CALCULUS
EXPLORATION OF THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

\[
\frac{d}{dx} \int_1^x t^2 \, dt = \\
\frac{d}{dx} \int_{\pi/6}^x \cos t \, dt = \\n\frac{d}{dx} \int_2^4 t^2 \, dt = \\
\frac{d}{dx} \int_{\pi/6}^{\pi^2} \cos t \, dt = 
\]

**Second Fundamental Theorem of Calculus:**

\[
\frac{d}{dx} \int_a^x f(t) \, dt = 
\]

\[
\frac{d}{dx} \int_x^a f(t) \, dt = 
\]

**Second Fundamental Theorem of Calculus (Chain Rule Version):**

\[
\frac{d}{dx} \int_a^x f(t) \, dt = 
\]

Ex. Use the Second Fundamental Theorem to evaluate:

(a) \[
\frac{d}{dx} \int_3^x \sqrt{1+t^2} \, dt =
\]

(b) \[
\frac{d}{dx} \int_2^x \tan t^3 \, dt =
\]

(c) \[
\frac{d}{dx} \int_{-1}^{x^3} \frac{1}{1+t} \, dt =
\]

(d) \[
\frac{d}{dx} \int_2^{\sin x} \sqrt{1+t^2} \, dt =
\]


The graph of a function $f$ consists of a quarter circle and line segments. Let $g$ be the function given by

$$ g(x) = \int_0^x f(t) \, dt. $$

(a) Find $g(0), g(-1), g(2), g(5)$.

(b) Find all values of $x$ on the open interval $-1, 5$ at which $g$ has a relative maximum. Justify your answer.

(c) Find the absolute minimum value of $g$ on $-1, 5$ and the value of $x$ at which it occurs. Justify your answer.

(d) Find the $x$-coordinate of each point of inflection of the graph of $g$ on $-1, 5$. Justify your answer.
CALCULUS
WORKSHEET ON SECOND FUNDAMENTAL THEOREM
AND FUNCTIONS DEFINED BY INTEGRALS

1. Find the derivatives of the functions defined by the following integrals:
   (a) \( \int_0^x \frac{\sin t}{t} \, dt \)
   (b) \( \int_0^x e^{-t^2} \, dt \)
   (c) \( \int_1^x \frac{\cos t}{t} \, dt \)
   (d) \( \int_0^1 e^{\tan^2 t} \, dt \)
   (e) \( \int_0^x \frac{1}{2t} \, dt \), \( x > 0 \)
   (f) \( \int_x^2 \cos t^2 \, dt \)
   (g) \( \int_1^x \frac{s^2}{s^2 + 1} \, ds \)
   (h) \( \int_5^{\cos x} t \cos t^3 \, dt \)
   (i) \( \int_{\tan x}^{\pi} \sin t^4 \, dt \)

2. The graph of a function \( f \) consists of a semicircle and two line segments as shown.
   Let \( g \) be the function given by
   \[ g(x) = \int_0^x f(t) \, dt. \]
   (a) Find \( g(0), g(3), g(-2), \) and \( g(5) \).
   (b) Find all values of \( x \) on the open interval \(-2,5\) at which \( g \) has a relative maximum.
      Justify your answers.
   (c) Find the absolute minimum value of \( g \) on the closed interval \([-2,5]\) and the value of \( x \) at which it occurs. Justify your answer.
   (d) Write an equation for the line tangent to the graph of \( g \) at \( x = 3 \).
   (e) Find the \( x \)-coordinate of each point of inflection of the graph of \( g \) on the open interval \(-2,5\). Justify your answer.
   (f) Find the range of \( g \).
3. Let \( g(x) = \int_0^x f(t) \, dt \), where \( f \) is the function whose graph is shown.
   (a) Evaluate \( g(0), g(1), g(2), \) and \( g(6) \).

   (b) On what intervals is \( g \) increasing?

   (c) Where does \( g \) have a maximum value? What is the maximum value?

   (d) Where does \( g \) have a minimum value? What is the minimum value?

   (e) Sketch a rough graph of \( g \) on \([0, 7]\).

4. Let \( g(x) = \int_{-3}^x f(t) \, dt \), where \( f \) is the function whose graph is shown.
   (a) Evaluate \( g(-3) \) and \( g(3) \).

   (b) At what values of \( x \) is \( g \) increasing? Justify.

   (c) At what values of \( x \) does \( g \) have a maximum value? Justify.

   (d) At what values of \( x \) does \( g \) have a minimum value? Justify.

   (e) At what values of \( x \) does \( g \) have an inflection point? Justify.
5. Use the function \( f \) in the figure and the function \( g \) defined by

\[
g(x) = \int_0^x f(t) \, dt.
\]

(a) Complete the table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Plot the points from the table in part (a).

(c) Where does \( g \) have its minimum?

(d) Which four consecutive points are collinear?

(e) Between which two consecutive points does \( g \) increase at the greatest rate?

5. If \( F(x) = \int_0^x f(t) \, dt \)

   a. Identify all critical numbers of \( F(x) \).

   b. On what interval(s) is \( F(x) \) decreasing?

   c. On what interval(s) is \( F(x) \) concave up?
CALCULUS
WORKSHEET 2 ON FUNCTIONS DEFINED BY INTEGRALS

1. Find the equation of the tangent line to the curve \( y = F(x) \) where \( F(x) = \int_0^x \sqrt{t^2 + 7} \, dt \) at the point on the curve where \( x = 1 \).

2. Suppose that \( 5x^3 + 40 = \int_c^x f(t) \, dt \).
   (a) What is \( f(x) \)?
   (b) Find the value of \( c \).

3. If \( F(x) = \int_{-4}^x t - 1\sqrt{t + 3} \, dt \), for what values of \( x \) is \( F \) decreasing? Justify your answer.

4. Let \( H(x) = \int_0^x f(t) \, dt \) where \( f \) is the continuous function with domain \([0, 12]\) shown on the right.
   (a) Find \( H(0) \).
   (b) On what interval(s) of \( x \) is \( H \) increasing? Justify your answer.
   (c) On what interval(s) of \( x \) is \( H \) concave up? Justify your answer.
   (d) Is \( H(12) \) positive or negative? Explain.
   (e) For what value of \( x \) does \( H \) achieve its maximum value? Explain.
5. The graph of a function $f$ consists of a semicircle and two line segments as shown on the right.

Let $g(x) = \int_1^x f(t) \, dt$.

(a) Find $g(1)$, $g(3)$, $g(-1)$.

(b) On what interval(s) of $x$ is $g$ decreasing? Justify your answer.

(c) Find all values of $x$ on the open interval $-3, 4$ at which $g$ has a relative minimum. Justify your answer.

(d) Find the absolute maximum value of $g$ on the interval $-3, 4$ and the value of $x$ at which it occurs. Justify your answer.

(e) On what interval(s) of $x$ is $g$ concave up? Justify your answer.

(f) For what value(s) of $x$ does the graph of $g$ have an inflection point? Justify your answer.

(g) Write an equation for the line tangent to the graph of $g$ at $x = -1$.

6. The graph of the function $f$, consisting of three line segments, is shown on the right.

Let $g(x) = \int_1^x f(t) \, dt$.

(a) Find $g(2)$, $g(4)$, $g(-2)$.

(b) Find $g'(0)$ and $g'(3)$.

(c) Find the instantaneous rate of change of $g$ with respect to $x$ at $x = 2$.

(d) Find the absolute maximum value of $g$ on the interval $-2, 4$. Justify your answer.

(e) The second derivative of $g$ is not defined at $x = 1$ and at $x = 2$. Which of these values are $x$-coordinates of points of inflection of the graph of $g$? Justify your answer.
CALCULUS
WORKSHEET 3 ON FUNCTIONS DEFINED BY INTEGRALS

Work the following on notebook paper.

1. The function $g$ is defined on the interval $[0, 6]$ by
   \[ g(x) = \int_0^x f(t) \, dt \]
   where $f$ is the function graphed in the figure.
   (a) For what values of $x$, $0 < x < 6$, does $g$ have a relative maximum? Justify your answer.
   
   (b) For what values of $x$ is the graph of $g$ concave down? Justify your answer.
   
   (c) Write an equation for the tangent line to $g$ at the point where $x = 3$.
   
   (d) Sketch a graph of the function $g$. List the coordinates of all critical point and inflection points.

2. Suppose that $f'$ is a continuous function, that $f(1) = 13$, and that $f(10) = 7$. Find the average value of $f'$ over the interval $[1, 10]$.

3. The graph of a differentiable function $f$ on the closed interval $[-4, 4]$ is shown.
   Let $G(x) = \int_{-4}^x f(t) \, dt$ for $-4 \leq x \leq 4$.
   (a) Find $G(-4)$.
   (b) Find $G(-4)$.
   (c) On which interval or intervals is the graph of $G$ decreasing? Justify your answer.
   
   (d) On which interval or intervals is the graph of $G$ concave down? Justify your answer.
   
   (e) For what values of $x$ does $G$ have an inflection point? Justify your answer.
4. The function \( F \) is defined for all \( x \) by \( F(x) = \int_0^x \sqrt{t^2 + 8} \, dt \).

(a) Find \( F'(x) \).

(b) Find \( F''(1) \).

(c) Find \( F''(x) \).

(d) Find \( F''(1) \).

5. If \( F(x) = \int_x^{-5} t^2 - t - 6 \, dt \), on what intervals is \( F \) decreasing?

6. The graph of the velocity \( v(t) \), in ft/sec, of a car traveling on a straight road, for \( 0 \leq t \leq 35 \), is shown in the figure.

(a) Find the average acceleration of the car, in \( \text{ft/ sec}^2 \), over the interval \( 0 \leq t \leq 35 \).

(b) Find an approximation for the acceleration of the car, in \( \text{ft/ sec}^2 \), at \( t = 20 \). Show your computations.

(c) Approximate \( \int_5^{35} v(t) \, dt \) with a Riemann sum, using the midpoints of three subintervals of equal length. Explain the meaning of this integral.
7. The function \( F \) is defined for all \( x \) by \( F(x) = \int_0^x f(t) \, dt \),

where \( f \) is the function graphed in the figure. The graph of \( f \) is made up of straight lines and a semicircle.

(a) For what values of \( x \) is \( F \) decreasing? Justify your answer.

(b) For what values of \( x \) does \( F \) have a local maximum? A local minimum? Justify your answer.

(c) Evaluate \( F(2) \), \( F'(2) \), and \( F''(2) \).

(d) Write an equation of the line tangent to the graph of \( F \) at \( x = 4 \).

(e) For what values of \( x \) does \( F \) have an inflection point? Justify your answer.
Answers to Worksheets on Second Fund. Th. & Functions Defined by Integrals

1. (a) \( \frac{\sin x}{x} \)
   
   (b) \( e^{-x^2} \)
   
   (c) - tan x
   
   (d) 0
   
   (e) \( \frac{1}{2x} \)
   
   (f) -cos \ x^2
   
   (g) \( \frac{x}{2\sqrt{x} \cdot x+1} \)
   
   (h) \( -\sin x \cos x \cos \cos^3 x \)
   
   (i) \( -\sin \tan^4 x \sec^2 x \)

2. (a) 0, \( \pi - \frac{1}{2}, -\pi, \pi - \frac{1}{2} \)
   
   (b) \( g \) has a rel. max. at \( x = 2 \) because \( g' \ x = f \ x \) changes from positive to negative there.
   
   (c) Abs. min. = \( -\pi \) at \( x = -2 \) (Justify with Candidates’ Test.)
   
   (d) \( y - \left( \pi - \frac{1}{2} \right) = - x - 3 \)
   
   (e) \( g \) has an I.P at \( x = 0 \) because \( g' \) changes from increasing to decreasing there.
      \( g \) has an I.P at \( x = 3 \) because \( g' \) changes from decreasing to increasing there.
   
   (f) \( [-\pi, \pi] \)

3. (a) 0, 2, 5, 3
   
   (b) \( g \) is increasing on (0, 3) since \( g' \) is positive there.
   
   (c) Max. value = 7 at \( x = 3 \) (Justify with Candidates’ Test.)
   
   (d) Min. value = 0 at \( x = 0 \) (Justify with Candidates’ Test.)
   
   (e) \( y - \frac{13}{2} = - x - 4 \)

4. (a) \( g \) is decreasing on \( \left( 1, \frac{3}{2} \right) \) and \( (4, 5) \) because \( g' \ x = f \ x \) is negative there.
   
   (b) \( g \) has a rel. max. at \( x = 1 \) and at \( x = 4 \) because \( g' \ x = f \ x \) changes from positive to negative there.
   
   (c) \( g \) is concave down on \( \left( \frac{1}{2}, 1 \frac{3}{4} \right) \) because \( g' \ x = f \ x \) is decreasing there.
   
   (d) \( g \) has an I.P at \( x = \frac{1}{2}, x = 1 \frac{3}{4}, \) and \( x = 3 \frac{1}{4} \) because \( g' \) changes from increasing to decreasing or vice versa there.
Worksheet 2 on Functions Defined by Integrals

1. $y = 2x - 2$

2. (a) $15x^2$  
   (b) $-2$

3. $F$ is decreasing on $x < -3$ because $F' x < 0$ there.

4. (a) 0
   (b) $H$ is increasing on $(0, 6)$ because $H' x = f' x$ is positive there.
   (c) $H$ is concave up on $(9.5, 12)$ because $H' x = f' x$ is increasing there.
   (d) $H$ is positive because there is more area above the $x$-axis than below.
   (e) $H$ achieves its maximum value at $x = 6$ because $H' 0 = 0$ and $H' 6$ and $H' 12$ are positive and $H' 6 > H' 12$.

5. (a) 0, $-1$, $-\pi$
   (b) $g$ is decreasing on $(1, 3)$ because $g' x = f' x$ is negative there.
   (c) $g$ has a relative minimum at $x = 3$ because $g' x = f' x$ changes from negative to positive there.
   (d) Abs. max. = 0 at $x = 1$ (Justify with Candidates’ Test.)
   (e) $g$ is concave up on $-3, -1$ and $2, 4$ because $g' x = f' x$ is increasing there.
   (f) $g$ has an inflection point at $x = -1$ and $x = 2$ because $g' x = f' x$ changes from increasing to decreasing or vice versa there.
   (g) $y + \pi = 2x + 1$

6. (a) 0, 2, $-\frac{9}{2}$
   (b) 0
   (c) 1
   (d) Abs. max. $= \frac{5}{2}$ at $x = 3$ (Justify with Candidates’ Test.)
   (e) $g$ has an inflection point at $x = 1$ because $g' x = f' x$ changes from increasing to decreasing there. $g$ does not have an inflection point at $x = 2$ because $g' x = f' x$ is decreasing for $1 < x < 2$ and continues to decrease on $2 < x < 4$. 
Worksheet 3 on Functions Defined by Integrals

1. (a) $g$ has a rel. max. at $x = 2$ because $g' x$, which is $f x$, changes from positive to negative there.
   (b) $g$ is concave down on $(1, 3)$ and $(5, 6)$ because $g' x$, which is $f x$, is decreasing there.
   (c) $y - \frac{1}{2} = -x - 3$
   (d) graph

2. $-\frac{2}{3}$

3. (a) 0  (b) 2
   (c) $G$ is decreasing on $(1, 3)$ because $G' x$, which is $f x$, is negative there.
   (d) $G$ has a rel. min. at $x = 3$ because $G' x$, which is $f x$, changes from negative to positive there.
   (e) $G$ is concave down on $-4, -3$ and $-1, 2$ because $G' x$, which is $f x$, is decreasing there.
   (f) $G$ has an inflection point at $x = 3$, $x = -1$, and $x = 2$ because $G' x$, which is $f x$, changes from decreasing to increasing or vice versa there.

4. (a) $2x\sqrt{x^2 + 8}$
   (b) 6
   (c) $\frac{4x^4}{\sqrt{x^4 + 8}} + 2\sqrt{x^4 + 8}$
   (d) $7\frac{1}{3}$

5. $F$ is decreasing on $x < -2$ and $x > 3$ because $F'$ is negative there.

6. (a) $\frac{6}{7}$ ft/sec²
   (b) $-2$ ft/sec² (using $(20, 40)$ and $(25, 30)$ to estimate the slope)
   (c) $(10)(30) + (10)(40) + (10)(20) = 900$ ft.
       This integral represents the approximate distance in feet that the car has traveled from $t = 5$ seconds to $t = 35$ seconds.

7. (a) $F$ is decreasing on $-5, -3.5$ and $2, 5$ because $F' x$, which is $f x$, is negative there.
   (b) $F$ has a local minimum at $x = -3.5$ because $F' x$, which is $f x$, changes from negative to positive there. $F$ has a local maximum at $x = 2$ because $F' x$, which is $f x$, changes from positive to negative there.
   (c) $4, 0, -3$
   (d) $y - 0 = -2 x - 4$
   (e) $F$ has an inflection point at $x = -3$, $x = -2$, $x = 1$, and $x = 3$ because $F' x$, which is $f x$, changes from increasing to decreasing or vice versa there.