RELATIONSHIPS OF DERIVATIVES

First Derivative Test

Let c be a critical value of a function f that is continuous on an open interval containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum of f.

If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum of f.

If $f'(x)$ does not change sign at c, then $f(c)$ is neither a relative minimum nor a relative maximum.

Second Derivative Test

Let f be a function such that $f'(c)=0$ and the second derivative of f exists on an open interval containing c.

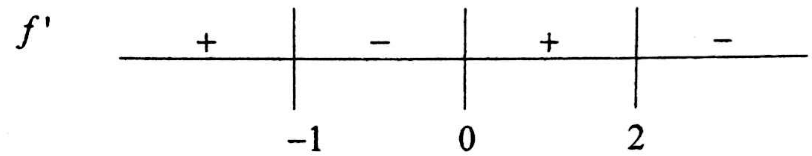
If $f''(c)>0$, then $f(c)$ is a relative minimum.

If $f''(c)<0$, then $f(c)$ is a relative maximum.

If $f''(c)=0$, the test fails. The First Derivative Test must be used.

I. LOCAL EXTREMA AND THE FIRST DERIVATIVE TEST

Let's consider a problem in which the student has been asked to find the local minimum of a continuous function and has summarized the behavior of the function using the sign chart below:



Using the First Derivative Test , the student should conclude that there is a local minimum at x = 0 because the first derivative changes from a negative value immediately to the left of the critical value to a positive value immediately to the right. Let's look at possible justification:

Not Sufficient:

f has a local minimum at $x = 0$.

f has a local minimum at $x = 0$ because f changes from decreasing to increasing.

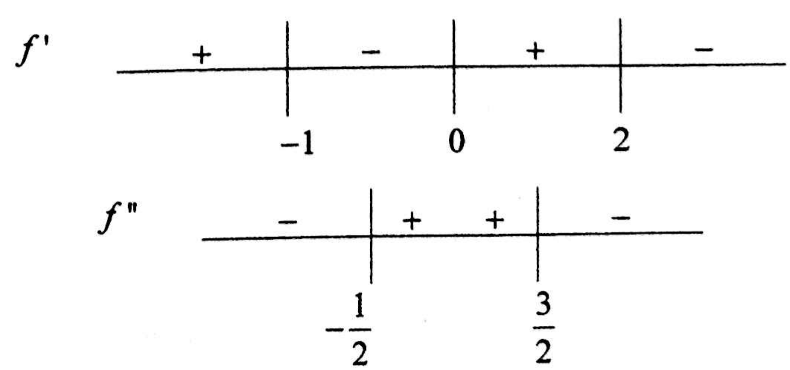
Sufficient:

f has a local minimum at $x = 0$ because f' changes from negative to positive.

f has a local minimum at $x = 0$ since f is decreasing to the left of $x = 0$ because f' is negative and f is increasing to the right because f' is positive.

II. LOCAL EXTREMA AND THE SECOND DERIVATIVE TEST

Let's look at the same situation, but assume the student will use the Second Derivative Test and has completed the following sign chart:



Using the Second Derivative Test, the student should conclude that there is a local minimum at the critical value x = 0 because the second derivative is positive.

Again, the sign chart is not considered a sufficient response for justification. The student needs to state what it is that leads to the conclusion:

Not Sufficient:

f has a local minimum at $x=0$.

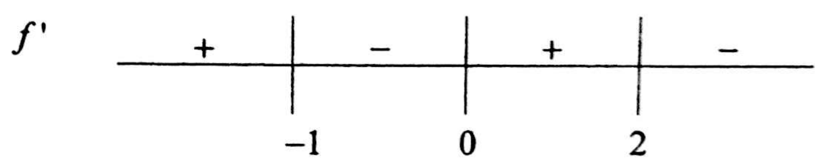
Sufficient:

f has a local minimum at $x=0$ because the first derivative is 0 and the second derivative is positive.

III. ABSOLUTE EXTREMA

On a closed interval, students must indicate that they have considered all critical values as well as the endpoints. This can be accomplished either by evaluating the function at each of these values, or by eliminating choices using the increasing/decreasing behavior of the function on the interval. In a graphical setting, students may also use an area comparison for justification.

If the student is asked to determine where the absolute minimum occurs for our function on the closed interval [-1, 2],



Not Sufficient:

The absolute is minimum occurs at $x=0$.

Sufficient:

$f(-1)=4$

$f(0)=-2$ The absolute minimum occurs at $x=0$.

$f(2)=3$

OR

Since f is decreasing on the interval $[-1,0]$ because f' is negative, and increasing on the interval $[0,2]$ because f' is positive, the absolute minimum occurs at $x=0$.

On an open interval, students must demonstrate that they have considered the entire interval even though the only possible choices are critical values. For example, on the open interval (-1,2),

Not Sufficient:

The absolute minimum occurs at $x=0$.

Sufficient:

Since $f'<0$ for all $x<0$ and $f'>0$ for all $x>0$, the absolute minimum occurs at $x=0$

Since $x=0$ is the only critical value on the interval $(-1,2)$, $f'\left(-\frac{1}{2}\right)=-1$ and $f'\left(\frac{1}{2}\right)=3$ the absolute minimum must occur at $x=0$.